

## Statistical mechanics of interfering links

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We consider the statistical mechanics of interfering transmissions in a wireless communications protocol. In this case, a connection between two nodes requires all other nodes within communication distance of the given two nodes to remain quiet on the given channel. This leads to an interesting problem of dimers on a lattice, with a restriction that no two dimers can overlap or be nearest neighbors. We consider both an equilibrium and a nonequilibrium, “greedy” dynamics for the links; the equilibrium properties of the model are found to exhibit an interesting spin-glass transition at maximum density on certain lattices, while the greedy construction is related to the problem of random sequential adsorption.

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The study of networks has identified a variety of interesting networks with different topological properties [1], with recent interest considering dynamical processes such as flows on the networks. Wireless communication networks provide a practically important example of information flow on a network [2,3]. In this paper, we consider the problem of the MAC (media access control) layer, which lies below the routing layer and allows different nodes access to given frequencies. This layer must enable a large number of nodes to communicate over a finite number of frequencies with finite signal range, while optimizing the latency, throughput, and fairness of the protocol [4]. We show that even the simplest model of the MAC layer gives rise to rich behavior, including glassy and nonequilibrium dynamics.

Recently, the problem of the MAC layer in such a protocol has been cast as a matching problem [5]. Consider some set of nodes, each with a given signal range. As a simple approximation, the signal range leads to a directed network of possible communication links between nodes: there is a link from node  $i$  to node  $j$  if node  $j$  can receive a transmission from node  $i$ . As a further simplification, we consider the case in which the network is *undirected*, so that whenever node  $j$  can receive node  $i$ , node  $i$  can also receive node  $j$ .

Suppose there is only one allowed frequency. In this case, suppose two nodes,  $i$  and  $j$  establish a communication link. Then, all nodes that are within range of node  $i$  or node  $j$  must remain silent, staying off the frequency in question, to avoid interference. Thus, it is possible for a given subset of links to be active simultaneously if and only if, for all nodes  $i$ , if node  $i$  is in an active link then  $i$  has exactly one neighbor  $j$  that is also active. Equivalently, this is the problem of finding a set of dimers in which nodes in different dimers are at least a distance of 2 from each other on the network; the presence of a dimer indicates an active link between the two nodes. The problem of multiple frequencies is similar, and leads to a coloring problem [6]. In Fig. 1, we show an example network and various possibilities of both allowed and disallowed sets of communication links.

In this paper, we study the statistical mechanics of this

problem. We consider two different ways to generate the links. In the first case, we consider summing over all possible allowed sets of dimers, weighting the covering by a fugacity for the number of dimers. By taking the fugacity to be large, we are thus able to find high-density configurations of dimers on the network. The ultimate goal would be to solve this problem on a random geometric graph [7]. In this paper, we begin this program by solving this problem on a Bethe lattice as a mean-field approximation to a graph with the given coordination number. Several surprising results are found even in this approximation. Interestingly, the results depend on the coordination number of the Bethe lattice: for a coordination number greater than 5, there appears to be a spin-glass transition at a finite fugacity, while for a coordination number less than 5, this transition is *fixed* at infinite fugacity. We also consider the problem on various finite dimensional lattices and speculate on the phase diagram. The problem is related to a recently considered statistical mechanics of dimers in the limit of infinite repulsion between neighboring dimers [8], but we identify the possibility of the spin-glass phase in this limit.

The second possibility we consider is a “greedy” dynamics, in which active links are added one at a time to the network, each link being added at random among the allowed possibilities. This dynamics is a version of random sequential adsorption [9]. We will see that the greedy dynamics does very poorly on graphs with a high coordination number.

To motivate the equilibrium and greedy dynamics, we start with the following more general set of dynamics: at

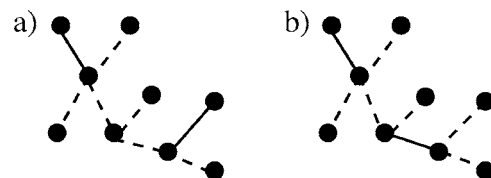


FIG. 1. (a) An example of an allowed configuration of dimers. The solid lines denote the presence of a dimer (active link), the dashed lines denote a link on the network that has no dimer (an inactive link). (b) A disallowed configuration of dimers. The two dimers are within a distance one of each other.

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every instant, we suppose that some set of dimers is present. We then suppose that dimers can be created, when a link is activated, and disappear, when two nodes that were communicating stop using the channel. Both processes happen with given rates as follows: We suppose that at every instant of time each link that is inactive has some rate of activating (if that does not interfere with other links), and we set this rate to unity. We assign each dimer some rate of disappearing in any instant of time, and we set this rate to  $1/\phi$ . Then, in the stationary distribution of this dynamics, the probability of finding any given allowed configuration of dimers is  $Z^{-1}\phi^{N_D}$ , where  $N_D$  is the number of dimers in the given configuration and

$$Z = \sum_c \phi^{N_D(c)}, \quad (1)$$

where  $c$  is a configuration of dimers and  $N_D(c)$  is the number of dimers in the configuration.

Then, the larger  $\phi$  is, the larger the average number of active links will be, which may be desired. However, a very large  $\phi$  means that the configuration of active links changes slowly in time; the system will tend to remain stuck in a given configuration for a long time, which is also undesirable. We consider two cases below. First, the stationary distribution at finite  $\phi$ . Next, we consider  $\phi = \infty$ , and start from an initial configuration with no links active; in this case, links can be activated but are never deactivated. This leads to a random sequential adsorption problem, and we will see that the final density of links in this case is far below the maximum possible in finite  $\phi$ .

*Equilibrium Distribution on the Bethe Lattice.* We start with the equilibrium case, solving for the partition function (1). This problem is equivalent to that studied in [8] in the limit  $\alpha \rightarrow -\infty$  (in that paper's notation). However, we consider the possibility of different spin-glass and ordering transitions in this paper.

We consider a Bethe lattice with fixed branching ratio,  $k$ , so that every node has  $q = k + 1$  neighbors. There are a number of different possible boundary conditions considered in the literature [10]. One possibility is to assume that the lattice indeed is a tree with a given branching ratio and with some number of leaves; in this case, appropriate boundary conditions must be assigned at the leaves of the tree and all the results given in this paper only apply to spins that are remote from the boundary. This procedure is reasonable in unfrustrated systems, and in general for systems outside of a spin-glass phase. However, frustrated systems in the spin-glass phase are sensitive to the boundary conditions that fix the degree of frustration [11]. Then, a better choice is a random graph with fixed connectivity  $k + 1$ .

We will solve the problem with a set of recurrence equations on the Bethe lattice, and identify the onset of a spin-glass phase. In the spin-glass phase, which we do not treat, the system becomes sensitive to boundary conditions. Outside the spin-glass phase, the simple solution with recurrence equations is adequate to describe both the case of a tree with random boundary conditions, considering nodes remote from the boundary, and the case of a random graph.

To obtain the recurrence equations, we consider a tree

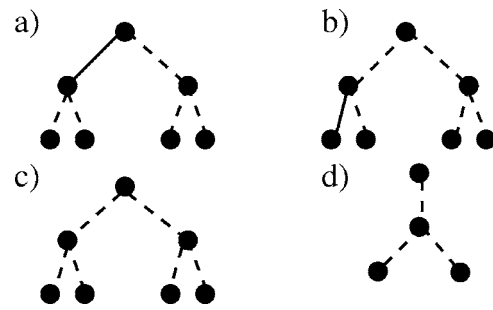


FIG. 2. (a) Active configuration: root connected to a daughter by an active link. (b) Locked configuration: a daughter involved in an active link, preventing the root from forming a link to any other node. (c) Free configuration: none of the daughters have active links. (d) Joining  $k + 1$  trees as described in text.

with one of three possible states for the root of the tree (the root has only  $k$  neighbors). First, the root can be connected to one of its daughters via a dimer (we call this active). Second, the root can be not connected to any of its daughters, but have at least one of its daughters involved in an active link (we refer to this as locked). Third, the root can be not connected to any of its daughters, and have none of its daughters active either (we refer to this as free). These three possibilities are shown in Figs. 2(a)–2(c).

Let the partition function of the dimer problem on a tree in each of the three cases be  $Z_1, Z_2, Z_3$ , respectively. Then take  $k$  of these trees and join them together to make a tree of one higher level, and compute the resulting partition functions  $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$  for the larger tree with the three different boundary conditions for that tree.

We have that  $\tilde{Z}_1 = \phi k Z_3 (Z_2 + Z_3)^{k-1}$ , as one of the  $k$  daughters must be free to have a dimer with the new root, and all the other daughters must be locked or free (they cannot be active or the root could not add a dimer to the free daughter). Then,  $\tilde{Z}_3 = (Z_2 + Z_3)^k$ , as each of the  $k$  daughters of the new root must be either locked or free, but cannot be active, so each subtree has partition function  $Z_2 + Z_3$ . Finally,  $\tilde{Z}_2 = (Z_1 + Z_2 + Z_3)^k - (Z_2 + Z_3)^k$ , as in this case at least one daughter must be active; we consider all possible states of the daughters,  $(Z_1 + Z_2 + Z_3)^k$ , and subtract off those states with no daughters active,  $\tilde{Z}_2 = (Z_2 + Z_3)^k$ .

It is convenient to rescale the partition functions  $Z_i$  by a constant in looking for a fixed point. We scale  $\tilde{Z}_i \rightarrow \tilde{Z}_i / \tilde{Z}_3$ , so  $\tilde{Z}_3 = 1$ . The rescaled equations become

$$\tilde{Z}_3 = 1; \quad \tilde{Z}_1 = \phi k / (Z_2 + 1);$$

$$\tilde{Z}_2 = (Z_1 + Z_2 + 1)^k / (Z_2 + 1)^k - 1. \quad (2)$$

We have numerically solved Eqs. (2) and found that for small  $\phi$  the solution converges to a fixed point, which we refer to as a liquid phase as we find below that it has no long-range correlations. For larger  $\phi$ , there is a range for which the fixed point is stable, but there is also a stable set of solutions that *oscillates* with period three. The oscillating

solution is referred to as a solid phase, as it has broken translational symmetry: at a given level of the tree most of the nodes are free, at the next higher level most of the nodes are active to a daughter, and at the next higher level most of the nodes are locked.

At even higher values of  $\phi$ , the stationary solution becomes unstable to oscillations and only the solid solution remains. The existence of the solid solutions depends on the boundary conditions on the tree. If the lattice is a balanced tree with a given branching ratio and depth, then the system has only two phases, solid and liquid, with a phase transition between them (the possibility of a liquid-gas transition as in [8] does not happen in this system) as identified from the recurrence equations above. However, on a random graph, the presence of loops will frustrate the solid phase. The solid phase is also frustrated on a tree in which the boundary conditions on the leaves are chosen sufficiently randomly (for example, the tree is not balanced, so that different leaves are at different depths below the root). While these frustrating boundary conditions can destroy the solid phase, they raise the possibility of a spin-glass phase, considered below.

At large  $\phi$ , we can solve Eq. (2) in the liquid phase giving

$$Z_1^{liquid} = k^{(k+1)/(2k+1)} \phi^{(k+1)/(2k+1)},$$

$$Z_2^{liquid} = k^{k/(2k+1)} \phi^{k/(2k+1)}, Z_3^{liquid} = 1.$$

To find the probability that a given site has a dimer leaving it, we must then connect  $q=k+1$  of these trees into one tree, as in Fig. 2(d), rather than  $k$  trees as above. After carrying this out, one finds that for large  $\phi$  a site has probability  $(k+1)/(2k+1) - \mathcal{O}(\phi^{-1/(2k+1)})$  of having an active link emanating from it. Defining  $N$  to be the total number of sites and  $\rho = 2N_D/N$ , we get a limiting density of active sites at large  $\phi$  of  $\rho = (k+1)/(2k+1) = q/(2q-1)$ .

The limiting density in the liquid phase can be understood on physical grounds. For *any* lattice with fixed coordination number  $k+1$  and for which the number of sites that neighbor a dimer is fixed to be  $n$  (if there are no loops of length 3,  $n=2k$ ), the density of active sites is at most  $(k+1)/(k+1+n/2)$ . To see this, note that the total number of sites that neighbor dimers is at least  $N_D n/(k+1)$ : each of the  $N_D$  dimers has  $n$  neighbors, but since each of those neighboring sites has  $k+1$  neighbors, we may have overcounted the number of sites that neighbor dimers by a factor  $k+1$ . Then, the total number of sites that have active links, plus the total number of sites that neighbor a site with an active link is at least  $N_D [2+n/(k+1)] \leq N$ . Thus,  $\rho = 2N_D/N \leq 2/[2+n/(k+1)] = (k+1)/(k+1+n/2)$ . For a lattice with no loops of length three, this reduces to  $\rho \leq (k+1)/(2k+1)$ ; to achieve the maximum density, it is necessary that if a node does not have an active link then all  $k+1$  of its neighbors have active links as in Fig. 3(a). The limiting density in the liquid phase saturates this bound, but we now consider the possibility of a spin-glass transition in the system that may prevent it from reaching the maximum density.

To identify the spin-glass transition, we follow the procedure used for the Ising spin glass [12,10]. Let each of the  $k$  trees that we join together in the procedure above have slight random variations in the values of  $Z_1, Z_2, Z_3$  about the liquid

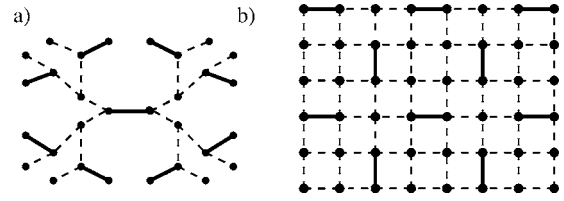


FIG. 3. (a) Configuration to obtain highest density on Bethe lattice. (b) Highest possible density configuration on the square lattice.

solution above, resulting from randomness in the boundary conditions. If the liquid solution is stable against these random fluctuations, then the Bethe lattice and random graph will behave identically in the thermodynamic limit. Otherwise, we go to a spin-glass phase, and the complete graph may have different behavior. Fixing  $Z_3=1$  as above, we have two physical quantities,  $Z_1$  and  $Z_2$  for each subtree. Suppose first that on each subtree each of these two values has *the same* slight fluctuation about the liquid solution above. We can compute the change in  $\tilde{Z}_1$  and  $\tilde{Z}_2$  to linear order from Eq. (2). The resulting linear transformation is found to have eigenvalues  $\lambda^\pm = -(k \pm \sqrt{k^2 - 4k})/2$  at  $\phi = \infty$ . Let these eigenvalues have left eigenvectors  $v^\pm$ . Following [12], we consider the mean-square fluctuations

$$M^\pm \equiv \overline{|\sum_i v_i^\pm (Z_i - Z_i^{liquid})|^2}$$

. Here, the overline denotes averaging over the random fluctuations, while the sum over  $i=1,2$  projects  $(Z_i - Z_i^{liquid})$  onto the corresponding right eigenvector. If on each subtree the fluctuations about the mean are random, then the mean-square fluctuation at the next higher level is given by  $\tilde{M}^\pm = |\lambda^\pm|^2 / k M^\pm$ , where the reduction by  $k$  is caused by averaging  $k$  random numbers together. When  $|\lambda|^2 = k$ , we have a spin-glass transition. Now, for  $k < 4$ , the two eigenvalues are complex conjugates with absolute value  $\sqrt{k}$  at  $\phi = \infty$ , so that the spin-glass transition occurs precisely at  $\phi = \infty$ , with no transition at  $\phi < \infty$ .

For  $k > 4$ , we have a spin-glass transition at finite  $\phi$ , with both eigenvalues real and only one of them greater than  $\sqrt{k}$  in absolute value. Since there is only one such such eigenvalue, this transition might be in the same universality class as the Ising spin glass on the random graph. One caveat is that we have identified an instability of the liquid phase, but have not yet shown the existence of solutions with a distribution of local partition functions  $Z_i$ ; also, any possible first-order transition to a spin-glass phase would be missed by this analysis.

*Equilibrium in Finite Dimensions.* The treatment above is on the Bethe lattice. The properties of the system on finite dimensional systems are also of interest. On a square lattice, the system has a number of possible ordered structures. The pattern of Fig. 3(b) is the highest possible density, with a density of  $\rho = 4/9$ , not far below the bound of  $4/7$  found above for a graph with a coordination number of 4. Thus, we

conjecture that as a function of  $\phi$  the system has a phase transition from a liquid state at low  $\phi$  to the ordered state of Fig. 3(b) at large  $\phi$ , with possible additional transitions to other ordered phases. A very interesting question is whether the system may have a glassy phase at some intermediate  $\phi$ .

*Greedy Dynamics.* The second problem we consider is the random sequential adsorption case at  $\phi=\infty$ . Our technique follows that of [13] in a slightly different case of no interaction between dimers. We start at time  $t=0$  with no active links on the lattice. We then activate allowed links at a unit rate per link. We consider the probability  $E_m(t)$  that after time  $t$  a connected cluster of  $m$  sites has: (1) no dimers on any of the  $m$  sites and (2) none of the neighbors of the given  $m$  sites has a dimer. These conditions imply that the  $m$  sites are free to participate in dimers as they are not blocked by any of their neighbors. It is interesting that the probability can be written as  $E_m(t)$ , independent of the structure of the connected cluster (there are many possible clusters with  $m$  sites). At  $t=0$ , we have  $E_m(t)=1$  for all  $m$ . At later times, one can derive the following differential equation for  $m>0$ :

$$\partial_t E_m = -(m-1)E_m - [(k-1)m+2]E_{m+1} - [(k-1)m+2]kE_{m+2}. \quad (3)$$

There, the first term corresponds to the possibility of adding a dimer to any of the  $(m-1)$  bonds that connect two sites, both in the cluster. The second term corresponds to the possibility of connecting a dimer to any of the  $[(k-1)m+2]$  bonds that connect a site in the cluster to a site outside the cluster, while the last term corresponds to the possibility that a site that neighbors the cluster will add a dimer to one of the  $k$  sites outside the cluster that neighbor the given site.

The ansatz  $E_m(t)=c(t)\sigma(t)^{m-1}$  solves Eq. (3) with  $\partial_t \sigma = -\sigma - (k-1)\sigma^2 - (k-1)k\sigma^3$  and  $\partial_t c = -(k+1)\sigma c - (k+1)k\sigma^2 c$ , and initial conditions  $c(0)=\sigma(0)=1$ .

We define  $\rho(t)=2N_D(t)/N$  to be the fraction of sites with a dimer emanating from them. We find  $\partial_t \rho = (k+1)E_2$ .

Numerical solution of the differential equation gives  $\rho(t \rightarrow \infty) \approx 0.313$  for  $k=4$ . This is noticeably worse than the limiting value of  $\rho=5/9$  computed in the equilibrium case above. At larger  $k$ , the nonequilibrium dynamics does even worse. The differential equations can be solved asymptotically at large  $k$ , where for  $c, \sigma \gg 1/k$  the equations simplify to  $\partial_t \sigma = -k^2 \sigma^3$ ,  $\partial_t c = -k^2 \sigma^2 c$ . The result for large  $k$  is  $\rho(t \rightarrow \infty) \sim \ln(k)/k$ .

*Discussion.* We have studied a simple model inspired by recent developments in wireless communication. This model turns out to be a model of interacting dimers very similar to other models studied in statistical physics. From the point of view of statistical physics, one of the most interesting features of the model is the spin-glass transition at  $\phi=\infty$  for  $k \leq 4$ . In the language of statistical physics, this implies that the transition happens at zero temperature in the mean-field limit. This may simplify the treatment of this transition and may be useful for studying finite dimensional spin glasses.

The spin-glass transition is of interest for practical purposes also. In the liquid phase, heuristic algorithms will likely do a good job of rapidly finding allowed dimer configurations of a given density, but in the glass phase it will be much more difficult to find good configurations and the system may get stuck in certain configurations.

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